

MONOMIAL BASES FOR QUANTUM AFFINE \mathfrak{sl}_n

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ABSTRACT. We use the idea of generic extensions to investigate the correspondence between the isomorphism classes of nilpotent representations of a cyclic quiver and the orbits in the corresponding representation varieties. We endow the set \mathcal{M} of such isoclasses with a monoid structure and identify the submonoid \mathcal{M}_c generated by simple modules. On the other hand, we use the partial ordering on the orbits (i.e., the Bruhat-Chevalley type ordering) to induce a poset structure on \mathcal{M} and describe the poset ideals generated by an element of the submonoid \mathcal{M}_c in terms of the existence of a certain composition series of the corresponding module. As applications of these results, we generalize some results of Ringel involving special words to results with no restriction on words and obtain a systematic description of many monomial bases for any given quantum affine \mathfrak{sl}_n .

1. INTRODUCTION

Let \mathbf{U} be a quantum group over $\mathbb{Q}(v)$ associated to a Cartan datum (I, \cdot) in the sense of [9, 1.1], and let $E_i, F_i, K_i^{\pm 1}$ ($i \in I$) be its generators. Then all monomials in the E_i 's, F_i 's and $K_i^{\pm 1}$'s span \mathbf{U} . It is natural to ask which monomials form bases for \mathbf{U} . Since \mathbf{U} admits a triangular decomposition

$$\mathbf{U} = \mathbf{U}^- \mathbf{U}^0 \mathbf{U}^+$$

where \mathbf{U}^+ (resp. \mathbf{U}^- , \mathbf{U}^0) is the subalgebra generated by the E_i 's (resp. F_i 's, $K_i^{\pm 1}$'s), and the monomials $K_{\mathbf{a}} = K_1^{a_1} \cdots K_n^{a_n}$ with $\mathbf{a} = \sum_i a_i i \in \mathbb{Z}I$ form a basis for \mathbf{U}^0 , it would be interesting to know monomial bases for \mathbf{U}^+ (and hence for \mathbf{U}^-). More precisely, let Ω be the set of words on the alphabet I . For each word $w = i_1 \cdots i_m \in \Omega$, let

$$E_w = E_{i_1} \cdots E_{i_m}, \quad F_w = F_{i_1} \cdots F_{i_m}.$$

We are interested in finding subsets $\Omega' \subset \Omega$ such that the set $\{E_w\}_{w \in \Omega'}$ forms a basis for \mathbf{U}^+ .

In the case where (I, \cdot) is of simply-laced finite type, Lusztig introduced certain monomial bases for \mathbf{U}^+ in [7, 7.8] relative to a fixed reduced expression of the longest word in the corresponding Weyl group. Interesting applications of monomial bases can be found in, e.g., [15], [2], [11] and [3]. In [3] it is proved that some (integral) monomial bases for a quantum \mathfrak{gl}_n give rise to monomial bases for q -Schur algebras and Hecke algebras (see, e.g., [3, 7.2, 9.4]).

In the investigation [13] of realizing the generic composition algebra of a cyclic quiver as the +part of a quantum affine \mathfrak{sl}_n , Ringel constructed certain monomial bases over some

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so-called condensed words whose definition is rather long and complicated. In this paper, we shall describe monomial bases for a quantum affine \mathfrak{sl}_n in a more general and satisfactory way. We shall prove the following monomial basis theorem (see §8).

Theorem 1.1. *Let $\mathbf{U} = \mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$. Then there is a partition $\Omega = \cup_{\pi \in \Pi^s} \Omega_\pi$ such that, for any subsets $\Omega^\pm \subset \Omega$ with $|\Omega^\pm \cap \Omega_\pi| = 1$ for every $\pi \in \Pi^s$, the set*

$$\{F_y K_{\mathbf{a}} E_w \mid y \in \Omega^-, w \in \Omega^+, \mathbf{a} \in \mathbb{Z}I\}$$

forms a basis for \mathbf{U} .

The proof of the theorem uses the idea of generic extensions of nilpotent representations of a cyclic quiver with n vertices. We first generalize a recent work by Reineke [10] to obtain a monoid structure on the set \mathcal{M} of isoclasses of nilpotent representations indexed by Π , the set of n -tuples of partitions. Simple modules will generate a submonoid \mathcal{M}_c . Thus, every word $w \in \Omega$ defines a unique isoclass in \mathcal{M}_c which is indexed by $\wp(w)$. We shall prove that the set of all $\wp(w)$ coincides with the subset Π^s of separated multipartitions defined in [13, 4.1]. Thus, the fibres of \wp yield a partition of Ω . Our argument will then depend heavily on the structure of nilpotent representations.

We organise the paper as follows. We start with investigating several useful properties of nilpotent representations in §2. Then we move on looking at generic extensions of nilpotent representations and constructing the monoid \mathcal{M} in §3. In §4, we discuss the relation between the submonoid \mathcal{M}_c and separated multipartitions. Some algorithms are introduced to calculate the multipartition $\wp(w)$ and the words in the fibres of \wp . The notion of distinguished words is defined in terms of a certain module structure. However a combinatorial criterion exists. These will be discussed in §5. There are two partial order relations on nilpotent representations defined geometrically by the inclusive relation on the closures of orbits and algebraically by module extensions. We shall prove in §6 that the two relations coincide. Moreover, we describe a poset ideal generated by an element of \mathcal{M}_c in terms of the existence of certain composition series. This is Theorem 6.3. Some applications are given in the last three sections. In §7, we generalize some results of Ringel given in [13], and in §8, we prove Theorem 1.1. Finally, in the last section, we construct a PBW type basis for \mathbf{U}^+ and speculate some relations between the various bases including the canonical basis.

In a forthcoming paper, we expect to prove a similar result for the quantum groups of simply-laced finite type.

Some notation. Throughout this paper, k denotes a field. We shall assume that $k = \bar{k}$ is algebraically closed in section three. All modules M are finite dimensional over k . We denote by $\text{rad}(M)$ the radical of M , i.e., the intersection of all maximal submodules of M , and by $\text{top}(M) := M/\text{rad}(M)$ the top of M .

The following lemma is a result of a pull-back or push-out diagram.

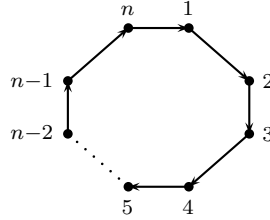
Lemma 1.2. *Let $0 \rightarrow X \rightarrow M \rightarrow Y \rightarrow 0$ be a short exact sequence of modules. Then any exact sequence $0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0$ gives rise to a commutative diagram with exact rows and columns*

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & M' & \longrightarrow & Y' \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \longrightarrow & X & \longrightarrow & M & \longrightarrow & Y \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & Y'' = Y'' & & \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

A similar result holds for an exact sequence $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$.

2. NILPOTENT REPRESENTATIONS OF A CYCLIC QUIVER

Let $\Delta = \Delta_n$ ($n \geq 2$) be the cyclic quiver with n vertices:



Then Δ determines the Cartan datum (I, \cdot) with $I = \{1, 2, \dots, n\}$ and $i \cdot i = 2$ and $2 \frac{i \cdot j}{i \cdot i} = -\delta_{j, i+1}$ ($i, j \in I$). It is always understood that I is identified with $\mathbb{Z}/n\mathbb{Z} = \{\bar{1}, \bar{2}, \dots, \bar{n}\}$. When no risk arises, we always drop the bars on the elements of I for notational simplicity. Thus, $n+1$ is understood as 1.

Let k be a field. A representation $V = (V_i, f_i)_i$ of Δ over k is called *nilpotent* if the composition $f_n \cdots f_2 f_1 : V_1 \rightarrow V_1$ is nilpotent or equivalently, one of the $f_{i-1} \cdots f_n f_1 \cdots f_i : V_i \rightarrow V_i$, $2 \leq i \leq n$, is nilpotent. The dimension vector of V is defined to be $\mathbf{dim} V = (\dim_k V_i)_i \in \mathbb{N}^n$. By $\mathbb{T} = \mathbb{T}(n, k)$ we denote the category of finite-dimensional nilpotent representations of Δ over k . Then \mathbb{T} is an abelian subcategory of $\text{Rep } \Delta$, the category of all finite-dimensional representations of Δ . Thus, the objects in \mathbb{T} are also called modules. Note that each nilpotent representation M with dimension d can be considered as a module over finite dimensional self-injective algebras $k\Delta/J^m$ for $m \geq d+1$, where J denotes the ideal of the path algebra $k\Delta$ generated by all arrows of Δ .

For each vertex $i \in I$, we have a one-dimensional representation (or simple module) $S_i = S_{ik}$. These S_i form a complete set of simple objects in \mathbb{T} . Further, for each integer $l \geq 1$, there is a unique (up to isomorphism) indecomposable representation $S_i[l]$ in \mathbb{T} of length l with top S_i . It is well known that $S_i[l]$, $i \in I, l \geq 1$, yield all isoclasses of indecomposable representations in \mathbb{T} . Then all isoclasses of representations in \mathbb{T} can be indexed by the set of n -tuples of partitions which we define in the following.

A partition is a sequence $p = (p_1, p_2, \dots, p_m, \dots)$ of non-negative integers in decreasing order

$$p_1 \geq p_2 \geq \dots \geq p_m \geq \dots$$

and containing only finitely many non-zero terms. The non-zero p_i are called the *parts* of p , and we say that the i th part is of *length* p_i . If $p_{m+1} = 0$, we will usually write

$p = (p_1, p_2, \dots, p_m)$. Thus we identify finite sequences which only differ by adding some zeros at the end. Note that the empty partition $\emptyset := (0, 0, \dots)$ is the only partition with 0 part. By \tilde{p} we denote the dual partition of p . Finally, we denote by Π the set of n -tuples of partitions. Then each n -tuple $\pi = (\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(n)})$ of partitions defines a representation

$$M(\pi) = M_k(\pi) = \bigoplus_{\substack{i \in I \\ j \geq 1}} S_i[\tilde{\pi}_j^{(i)}]$$

in $\mathbb{T}(n, k)$, where $\tilde{\pi}^{(i)} = (\tilde{\pi}_1^{(i)}, \tilde{\pi}_2^{(i)}, \dots)$ is the dual partition¹ of $\pi^{(i)}$, $i \in I$. In this way we obtain a bijection between Π and the set of isoclasses of representations in \mathbb{T} . Note that this bijection is independent of the field k .

For each module M in \mathbb{T} , we denote by $[M]$ the isoclass of M . For $a \geq 1$, we write

$$aM := \underbrace{M \oplus \dots \oplus M}_a.$$

For each partition $p = (p_1, p_2, \dots, p_m)$ and each $i \in I$, we set

$$M_i(p) = \bigoplus_{j=1}^m S_i[p_j].$$

Thus, if $\pi = (\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(n)}) \in \Pi$, we have

$$M(\pi) = \bigoplus_{i=1}^n M_i(\tilde{\pi}^{(i)}).$$

The following easy facts about nilpotent representations turn out to be very useful. For the convenience of the reader, we include some proof.

Lemma 2.1. *Let $p = (p_1, p_2, \dots, p_m)$ be a partition with m parts and assume $i \in I$.*

(1) *A representation L is an extension of S_i by $M_{i+1}(p)$ if and only if*

$$L \cong L_t := S_i[p_t + 1] \oplus \bigoplus_{j:j \neq t} S_{i+1}[p_j]$$

for some $1 \leq t \leq m+1$, where $p_{m+1} = 0$.

(2) *A submodule N of $M_i(p)$ is maximal if and only if*

$$N \cong N_t := S_{i+1}[p_t - 1] \oplus \bigoplus_{j:j \neq t} S_i[p_j]$$

for some $1 \leq t \leq m$.

(3) *If we choose $v_j \in S_i[p_j] \setminus \text{rad}(S_i[p_j])$ and $x_1, \dots, x_m \in k$ are not all zero, then the element $x_1 v_1 + \dots + x_m v_m \in M_i(p)$ generates a submodule isomorphic to $S_i[p_a]$, where a is the smallest index such that $x_a \neq 0$.*

¹Note that we adopt the notation from [13, 3.3] in order to view a module as a column in the Young diagram of a partition.

Proof. (1) Clearly, each L_t admits an exact sequence

$$0 \longrightarrow M_{i+1}(p) \longrightarrow L_t \longrightarrow S_i \longrightarrow 0,$$

that is, L_t is an extension of S_i by $M_{i+1}(p)$. Conversely, let L be any extension of S_i by $M_{i+1}(p)$. We may consider $M_{i+1}(p)$ as a submodule of L . Then

$$\text{rad}(M_{i+1}(p)) \subseteq \text{rad}(L) \subseteq M_{i+1}(p).$$

It follows that

$$\text{top}(L) = L/\text{rad}(L) \cong S_i \oplus M_{i+1}(p)/\text{rad}(L) \cong S_i \oplus aS_{i+1}$$

for some $a \geq 0$. Applying Krull-Remak-Schmidt theorem to L , we conclude that L is isomorphic to some L_t .

(2) If $N \cong N_t$ for some $1 \leq t \leq m$, it is obvious that N is a maximal submodule of $M := M_i(p)$. Conversely, let N be a maximal submodule of M . Then $\text{rad}(M) \subseteq N$ and there are exact sequences

$$0 \longrightarrow N/\text{rad}(M) \longrightarrow M/\text{rad}(M) \longrightarrow M/N \cong S_i \longrightarrow 0$$

and

$$0 \longrightarrow \text{rad}(M)/\text{rad}(N) \longrightarrow N/\text{rad}(N) \longrightarrow N/\text{rad}(M) \longrightarrow 0.$$

It follows that $N/\text{rad}(N) \cong (m-1)S_i \oplus aS_{i+1}$ for some $a \geq 0$. The statement (1) implies $a \leq 1$ since M is an extension of S_i by N . Hence, N is isomorphic to some N_t .

(3) This is obvious. \square

The next lemma describes extensions of indecomposable nilpotent representations by indecomposable ones.

Lemma 2.2. *Let $i, j \in I$ and $l, m \geq 1$. If there is a non-split exact sequence*

$$(2.2.1) \quad 0 \longrightarrow S_j[m] \longrightarrow M \longrightarrow S_i[l] \longrightarrow 0,$$

then there are integers $r \geq 1, s \geq 0$, and $t \geq 1$ such that $j = \overline{i+r}, l = r+s, m = s+t$ and $M \cong S_i[r+s+t] \oplus S_{\overline{i+r}}[s]$.

Proof. We apply induction on l . If $l = 1$, then $j = \overline{i+1}$. We take $r = 1, s = 0$, and $t = m$, as required. Let now $l > 1$. Consider the following exact sequence

$$0 \longrightarrow S_{\overline{i+1}}[l-1] \longrightarrow S_i[l] \longrightarrow S_i \longrightarrow 0.$$

This together with (2.2.1) induces by Lemma 1.2 exact sequences

$$(2.2.2) \quad 0 \rightarrow S_j[m] \rightarrow M' \rightarrow S_{\overline{i+1}}[l-1] \rightarrow 0$$

$$(2.2.3) \quad 0 \rightarrow M' \rightarrow M \rightarrow S_i \rightarrow 0.$$

If (2.2.2) splits and $j \neq \overline{i+1}$, then (2.2.1) splits, contrary to the assumption. Therefore, $j = \overline{i+1}$, $m \geq l$, and $M \cong S_i[m+1] \oplus S_{\overline{i+1}}[l-1]$. Thus, we can take $r = 1, s = l-1$ and $t = m-l+1$. If (2.2.2) does not split, then, by induction, there are integers $r' \geq 1, s' \geq 0$, and $t' \geq 1$ such that $j = \overline{i+1+r'}, l-1 = r'+s', m = s'+t'$ and

$$M' \cong S_{\overline{i+1}}[r'+s'+t'] \oplus S_{\overline{i+1+r'}}[s'].$$

By Lemma 2.1(1), (2.2.3) implies $M \cong S_i[r' + s' + t' + 1] \oplus S_{i+1+r'}[s']$. By putting $r = r' + 1, s = s'$ and $t = t'$, we get

$$M \cong S_i[r + s + t] \oplus S_{i+r}[s].$$

□

Extensions (2.2.1) can be used (cf. [13, 4.7]) to define a relation \prec on Π by setting $\mu \prec \lambda$ if there exist a $\pi \in \Pi$ and a non-split extension (2.2.1) such that $M(\lambda) \cong M(\pi) \oplus M$ and $M(\mu) \cong M(\pi) \oplus S_j[m] \oplus S_i[l]$. Thus, by Lemma 2.2, we see that $\mu \prec \lambda$ if and only if there are integers $r \geq 1, s \geq 0, t \geq 1, i \in I$, and $\pi \in \Pi$ such that (a) π is obtained from λ by deleting a part of length $r + s + t$ in $\tilde{\lambda}^{(i)}$ and a part of length s in $\tilde{\lambda}^{(i+r)}$, and (b) π is also obtained from μ by deleting a part of length $r + s$ in $\tilde{\mu}^{(i)}$ and another part of length $s + t$ in $\tilde{\mu}^{(i+r)}$. In other words, we have

$$\begin{aligned} M(\lambda) &= M(\pi) \oplus S_i[r + s + t] \oplus S_{i+r}[s] \\ \text{and } M(\mu) &= M(\pi) \oplus S_i[r + s] \oplus S_{i+r}[s + t]. \end{aligned}$$

In this case, there is an exact sequence

$$0 \longrightarrow S_{i+r}[s + t] \longrightarrow M(\lambda) \longrightarrow M(\pi) \oplus S_i[r + s] \longrightarrow 0.$$

Let \preceq be the partial ordering on Π generated by the relation \prec . Thus, $\mu \prec \lambda$ means that there are $\mu_0 = \mu, \mu_1, \dots, \mu_m = \lambda$ in Π such that $\mu_0 \prec \mu_1 \prec \dots \prec \mu_m$. We called \preceq the *extension order* on Π .

3. GENERIC EXTENSIONS AND THE MONOID \mathcal{M}

In this section k is algebraically closed. We study generic extensions of nilpotent representations. Their existence follows from a similar argument as in [10, Sect. 2].

Let $\mathbf{d} = (d_i)_i \in \mathbb{N}^n$. Then each representation $M = (k^{d_i}, f_i)_i$ (not necessarily nilpotent) of Δ can be identified with the point

$$f = (f_i)_i \in \prod_{i=1}^n \text{Hom}_k(k^{d_i}, k^{d_{i+1}}) \cong \prod_{i=1}^n k^{d_{i+1} \times d_i} =: R(\mathbf{d}),$$

where $d_{n+1} = d_1$. The algebraic group

$$GL(\mathbf{d}) = \prod_{i=1}^n GL_{d_i}(k)$$

acts on $R(\mathbf{d})$ by conjugation

$$(g_i)_i f = (g_{i+1} f_i g_i^{-1})_i,$$

where $g_{n+1} = g_1$. Then the $GL(\mathbf{d})$ -orbits in $R(\mathbf{d})$ correspond bijectively to the isoclasses of representations of Δ with dimension vector \mathbf{d} .

The stabilizer $GL(\mathbf{d})_M = \{g \in GL(\mathbf{d}) | gM = M\}$ of M is the automorphism group of M which is Zariski-open in $\text{End}(M)$ and has dimension equal to $\dim \text{End}(M)$. It follows that the orbit \mathcal{O}_M of M has dimension

$$\dim \mathcal{O}_M = \dim GL(\mathbf{d}) - \dim \text{End}(M).$$

For two representations M, N , we say that M degenerates to N , or that N is a *degeneration* of M , and write² $N \leq M$, if $\mathcal{O}_N \subseteq \overline{\mathcal{O}_M}$, the closure of \mathcal{O}_M . Note that $N < M \iff \mathcal{O}_N \subseteq \overline{\mathcal{O}_M} \setminus \mathcal{O}_M$. The following lemma is well-known (see for example [1, 1.1]).

Lemma 3.1. *If $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$ is exact and non-split, then $M \oplus N < E$.*

Since nilpotent representations with fixed dimension can be considered as modules over a finite dimensional representation-finite self-injective algebra $k\Delta/J^m$, [16, Thm 2] leads to the following alternative descriptions of degenerations of nilpotent representations.

Lemma 3.2. *Let M and N be nilpotent representations. Then the following statements are equivalent:*

- (1) $N \leq M$,
- (2) $\dim_k \text{Hom}(N, X) \geq \dim_k \text{Hom}(M, X)$ for all $X \in \mathbb{T}$,
- (3) $\dim_k \text{Hom}(X, N) \geq \dim_k \text{Hom}(X, M)$ for all $X \in \mathbb{T}$,
- (4) *there are representations M_i, X_i, Y_i ($1 \leq i \leq t$) in \mathbb{T} and short exact sequences $0 \rightarrow X_i \rightarrow M_i \rightarrow Y_i \rightarrow 0$ such that $M = M_1, M_{i+1} = X_i \oplus Y_i, 1 \leq i \leq t$, and $N = M_{t+1}$.*

We remark that, since the dimension $\dim_k \text{Hom}(X, Y)$ is independent of k for any $X, Y \in \mathbb{T}(n, k)$, the lemma guarantees that the ordering \leq on nilpotent representations is well-defined over any field k .

Further, let $R_0(\mathbf{d})$ be the subset of $R(\mathbf{d})$ consisting of nilpotent representations. It is easy to see that $R_0(\mathbf{d})$ is a closed subvariety of $R(\mathbf{d})$ stable under $GL(\mathbf{d})$ and contains only finitely many orbits. Clearly, an extension of a nilpotent representation by a nilpotent one is nilpotent. Hence, we have immediately the following (cf. [10, 2.1, 2.3]). We refer their proofs to [10].

Lemma 3.3. *For $\mathbf{d}, \mathbf{d}', \mathbf{d}'' \in \mathbb{N}^n$ with $\mathbf{d} = \mathbf{d}' + \mathbf{d}''$ and subsets $\mathcal{X} \subset R_0(\mathbf{d}')$ and $\mathcal{Y} \subset R_0(\mathbf{d}'')$ stable under $GL(\mathbf{d}')$ and $GL(\mathbf{d}'')$, respectively, let $\mathcal{E} = \mathcal{E}(\mathcal{X}, \mathcal{Y})$ be the subset of $R_0(\mathbf{d})$ of all extensions of representations $M \in \mathcal{X}$ by representations $N \in \mathcal{Y}$. Then \mathcal{E} is $GL(\mathbf{d})$ stable, and hence, constructible. Moreover, if \mathcal{X} and \mathcal{Y} are irreducible (resp. closed), then so is \mathcal{E} .*

Applying the lemma to $\mathcal{X} = \mathcal{O}_M$ and $\mathcal{Y} = \mathcal{O}_N$, we see that, for any two nilpotent representations M and N , there exists a unique extension G (up to isomorphism) of M by N with maximal dimension of \mathcal{O}_G , or equivalently, with minimal $\dim \text{End}(G)$. This representation G is called a *generic extension* of M by N and is denoted by $M * N := G$.

It can be seen easily that [10, 2.4] continues to hold in this case.

Proposition 3.4. *Let M, N, X be nilpotent representations of Δ . Then*

$$X \leq M * N \iff X \in \mathcal{E}(\mathcal{O}_{M'}, \mathcal{O}_{N'}) \text{ for some } M' \leq M, N' \leq N.$$

In particular, we have

$$M' \leq M, N' \leq N \implies M' * N' \leq M * N.$$

As a conclusion of Lemma 3.3 and Prop. 3.4, we can define a monoid $\mathcal{M} = \mathcal{M}(\Delta)$ whose elements are the isoclasses of nilpotent representations with multiplication $[M] * [N] = [M * N]$ and unit element $1_{\mathcal{M}} = [0]$, the isoclass of the zero representation. Note that

²The order \leq here is opposite to the so-called degeneration order used, e.g., in [10]. We follow a traditional notation for the ‘‘Bruhat-Chevalley order’’ of a Coxeter group.

the associativity of the multiplication $*$ follows from Lemma 1.2 and Prop. 3.4 (cf. [10, 3.1]). Inductively, we see easily that for any nilpotent representations M_1, M_2, \dots, M_t , the representation $M_1 * M_2 * \dots * M_t$ is the nilpotent representation G (unique up to isomorphism) with minimal dimension of its endomorphism algebra such that G has a filtration

$$G = G_0 \supset G_1 \supset \dots \supset G_{t-1} \supset G_t = 0$$

with $G_{s-1}/G_s \cong M_s$, $1 \leq s \leq t$.

Unlike the situation in [10], in our case the submonoid \mathcal{M}_c of \mathcal{M} generated by $[S_i]$, $i \in I$, is proper. However, we have the following similar result to [10, 3.4].

Proposition 3.5. (1) In case $n \geq 3$, the following relations hold in \mathcal{M}_c :

$$\begin{aligned} [S_i] * [S_j] &= [S_j] * [S_i] \text{ if } j \not\equiv i \pm 1 \pmod{n}, i, j \in I, \\ [S_i] * [S_{i+1}] * [S_i] &= [S_i] * [S_i] * [S_{i+1}], \\ [S_{i+1}] * [S_i] * [S_{i+1}] &= [S_i] * [S_{i+1}] * [S_{i+1}], i \in I. \end{aligned}$$

(2) In case $n = 2$, the following relations hold in \mathcal{M}_c :

$$\begin{aligned} [S_1] * [S_2] * [S_1] * [S_1] &= [S_1] * [S_1] * [S_2] * [S_1], \\ [S_2] * [S_1] * [S_2] * [S_2] &= [S_2] * [S_2] * [S_1] * [S_2]. \end{aligned}$$

The proof of this proposition is straightforward.

Our next purpose is to describe the generic extension of a simple representation by any nilpotent representation.

For each $i \in I$, we define a map

$$\sigma_i^+ : \Pi \rightarrow \Pi$$

as follows: for $\pi = (\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(n)}) \in \Pi$, $\sigma_i^+ \pi =: \lambda$ is defined by

$$\lambda^{(j)} = \begin{cases} \pi^{(j)} & \text{if } j \neq i, i+1 \\ \pi^{(i)} + \mathbf{1}_{\tilde{\pi}_1^{(i+1)}+1} & \text{if } j = i \\ \pi^{(i+1)} - \mathbf{1}_{\tilde{\pi}_1^{(i+1)}} & \text{if } j = i+1, \end{cases}$$

where $\mathbf{1}_a = (\underbrace{1, \dots, 1}_a)$. Note that, if we define λ dually, then $\tilde{\lambda}^{(i+1)}$ is obtained by deleting

the part $\tilde{\pi}_1^{(i+1)}$ from $\tilde{\pi}^{(i+1)}$, and $\tilde{\lambda}^{(i)}$ is obtained by adding the part $\tilde{\pi}_1^{(i+1)} + 1$ to $\tilde{\pi}^{(i)}$.

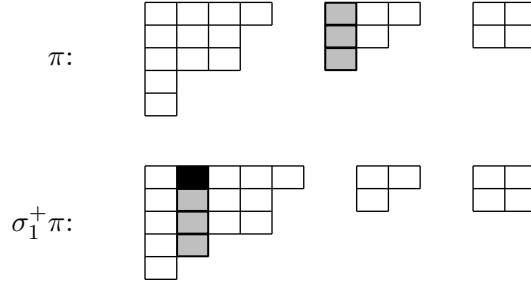
Example 3.6. Let $n = 3$ and $\pi = (\pi^{(1)}, \pi^{(2)}, \pi^{(3)})$ with

$$\pi^{(1)} = (4, 3, 3, 1, 1), \pi^{(2)} = (3, 2, 1), \pi^{(3)} = (2, 2).$$

It is easy to see that $\tilde{\pi}_1^{(1)} = 5, \tilde{\pi}_1^{(2)} = 3, \tilde{\pi}_1^{(3)} = 2$. Then

$$\begin{aligned} \sigma_1^+ \pi &= ((5, 4, 4, 2, 1), (2, 1), (2, 2)), \\ \sigma_2^+ \pi &= ((4, 3, 3, 1, 1), (4, 3, 2), (1, 1)), \\ \sigma_3^+ \pi &= ((3, 2, 2), (3, 2, 1), (3, 3, 1, 1, 1, 1)). \end{aligned}$$

Intuitively, for example, π and $\sigma_1^+ \pi$ are illustrated by their Young diagrams as follows:



Here the coloured column in $\sigma_1^+ \pi$ is obtained from the coloured one in π topped with the black box.

Proposition 3.7. *Let $i \in I$ and $\pi \in \Pi$. Then*

$$S_i * M(\pi) \cong M(\sigma_i^+ \pi).$$

Proof. Let $\pi = (\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(n)})$. For each $a \in I$, we set

$$M_a(\pi) := M_a(\tilde{\pi}^{(a)}) = \bigoplus_{j \geq 1} S_a[\tilde{\pi}_j^{(a)}],$$

where $\tilde{\pi}^{(a)} = (\tilde{\pi}_1^{(a)}, \tilde{\pi}_2^{(a)}, \dots)$ is the dual partition of $\pi^{(a)}$. Then we have $M(\pi) = \bigoplus_{a=1}^n M_a(\pi)$. Since $\text{Ext}^1(M_a(\pi), S_i) = 0$ for all $a \neq i+1$, it follows that

$$S_i * M(\pi) \cong \bigoplus_{a \neq i+1} M_a(\pi) \oplus (S_i * M_{i+1}(\pi)).$$

To prove the proposition, it suffices to prove

$$(3.7.1) \quad S_i * M_{i+1}(\pi) \cong S_i[\tilde{\pi}_1^{(i+1)} + 1] \oplus \bigoplus_{j \geq 2} S_{i+1}[\tilde{\pi}_j^{(i+1)}].$$

If $\pi^{(i+1)} = \emptyset$, this is obvious. Let now $\pi^{(i+1)} \neq \emptyset$. For simplicity, we write $\tilde{\pi}^{(i+1)} = (p_1, p_2, \dots, p_m)$ with $p_1 \geq p_2 \geq \dots \geq p_m > p_{m+1} = 0$, where $p_t = \tilde{\pi}_t^{(i+1)}$, $1 \leq t \leq m$. Then, by Lemma 2.1(1), each extension of S_i by $M_{i+1}(\pi)$ is isomorphic to

$$L_t = S_i[p_t + 1] \oplus \bigoplus_{j \neq t} S_{i+1}[p_j]$$

for some t with $1 \leq t \leq m+1$.

Suppose $1 \leq t \leq m-1$. If $p_t = p_{t+1}$, it is obvious that $L_t \cong L_{t+1}$. If $p_t > p_{t+1}$, we get a non-split exact sequence

$$0 \longrightarrow S_{i+1}[p_t] \longrightarrow S_i[p_t + 1] \oplus S_{i+1}[p_{t+1}] \longrightarrow S_i[p_{t+1} + 1] \longrightarrow 0.$$

This yields a non-split exact sequence

$$0 \longrightarrow \bigoplus_{j \neq t+1} S_{i+1}[p_j] \longrightarrow L_t \longrightarrow S_i[p_{t+1} + 1] \longrightarrow 0.$$

Hence, by Lemma 3.1

$$L_{t+1} = S_i[p_{t+1} + 1] \oplus \bigoplus_{j \neq t+1} S_{i+1}[p_j] < L_t.$$

Consequently, we have

$$S_i \oplus M_{i+1}(\pi) = L_{m+1} \leq L_m \leq \cdots \leq L_2 \leq L_1 \cong S_i * M_{i+1}(\pi),$$

proving (3.7.1). \square

Remark 3.8. (1) Dually, one can describe the generic extension of any nilpotent representation by a simple representation. We leave the detail to the reader.

(2) We observe from Lemma 2.1 that the number of non-isomorphic extensions of a simple module by any given nilpotent representation is independent of the field k . This together with the fact that $\dim \text{End}(M \oplus N) > \text{End}(E)$ for a non-split exact sequence $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$ in \mathbb{T} implies from the proof of 3.7 that, for any field k and $M_k \in \mathbb{T}(n, k)$,

$$(S_{ik} * M_k) \otimes \bar{k} \cong S_{i\bar{k}} * M_{\bar{k}},$$

where $S_{ik} * M_k$ is the extension of S_{ik} by M_k with minimal dimension of $\text{End}(S_{ik} * M_k)$.

4. THE SUBMONOID \mathcal{M}_c

Following [13, 4.1], an n -tuple $\pi = (\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(n)})$ of partitions is called *separated* if, for each $t \geq 1$, there is some $a(t) = a(t, \pi) \in I$ such that $\tilde{\pi}_j^{(a(t))} \neq t$ for all $j \geq 1$. By Π^s we denote the set of separated n -tuples of partitions. Then each σ_i^+ , $i \in I$ defined in §2 induces a map

$$\sigma_i^+ : \Pi^s \rightarrow \Pi^s.$$

Indeed, let $\pi \in \Pi^s$, for each $t \geq 1$, if $t \neq \tilde{\pi}_1^{(i+1)} + 1$, $(\widetilde{\sigma_i^+ \pi})^{(a(t, \pi))}$ has no part equal to t , and if $t = \tilde{\pi}_1^{(i+1)} + 1$, $(\sigma_i^+ \pi)^{(i+1)}$ has no part equal to t , proving $\sigma_i^+ \pi \in \Pi^s$.

We observe that, for $\pi = (\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(n)}) \in \Pi^s$, if $\pi \neq \emptyset$, the n -tuple of empty partitions, then $\tilde{\pi}_1^{(i)}$, $i \in I$, can not be the same, and there exists an $i \in I$ (not necessarily unique) such that $\tilde{\pi}_1^{(i)} > \tilde{\pi}_1^{(i+1)}$. (Note that $\tilde{\pi}^{(n+1)} = \tilde{\pi}^{(1)}$ and that $\tilde{\pi}_1^{(i+1)} = 0$ if $\tilde{\pi}^{(i+1)}$ is the empty partition.) Thus, if we put $\Pi_i^s = \sigma_i^+(\Pi^s)$, we have immediately from the definition that

$$\Pi^s = \{\emptyset\} \cup \bigcup_{i=1}^n \Pi_i^s.$$

As in [13, 2.1], let Ω denote the free semigroup with unit element, generated by the set $I = \{1, 2, \dots, n\}$. The elements in Ω are called *words* and are of the form $w = i_1 i_2 \cdots i_m$ with $i_1, i_2, \dots, i_m \in I$ and $m \geq 0$. For each $w = i_1 i_2 \cdots i_m \in \Omega$, we set

$$M(w) = S_{i_1} * S_{i_2} * \cdots * S_{i_m}.$$

Then there is a unique $\pi \in \Pi$ such that $M(w) \cong M(\pi)$, and we set $\wp(w) = \pi$. In this way we obtain a map

$$\wp : \Omega \rightarrow \Pi, w \mapsto \pi = \wp(w).$$

Note that by Remark 3.8(2) the map \wp is independent of the field k .

Theorem 4.1. (1) *The map \wp induces a surjection*

$$\wp : \Omega \twoheadrightarrow \Pi^s.$$

(2) *Let $\mathcal{M}(\Pi^s) := \{[M(\pi)] : \pi \in \Pi^s\}$. Then $\mathcal{M}_c = \mathcal{M}(\Pi^s)$.*

Proof. By Prop. 3.7 and induction on m , we see easily that, for $w = i_1 i_2 \cdots i_m \in \Omega$,

$$(4.1.1) \quad \wp(w) = \sigma_{i_1}^+ \sigma_{i_2}^+ \cdots \sigma_{i_m}^+ (\emptyset) \in \Pi^s,$$

since \emptyset is separated. Consequently, we obtain the inclusion $\mathcal{M}_c \subseteq \mathcal{M}(\Pi^s)$. Clearly, if \wp maps onto Π^s , then the inclusion $\mathcal{M}(\Pi^s) \subseteq \mathcal{M}_c$ follows. It remains to prove that \wp is surjective.

The surjective map $\sigma_i^+ : \Pi^s \rightarrow \Pi_i^s$ has a right inverse $\sigma_i^- : \Pi_i^s \rightarrow \Pi^s$ which can be defined as follow: for $\pi \in \Pi_i^s$, define $\mu := \sigma_i^-(\pi) \in \Pi$ by

$$(4.1.2) \quad \mu^{(a)} = \begin{cases} \mu^{(a)} & \text{if } a \neq i, i+1 \\ \mu^{(i)} - \mathbf{1}_{\tilde{\pi}_j^{(i)}} & \text{if } a = i \\ \mu^{(i+1)} + \mathbf{1}_{\tilde{\pi}_j^{(i)}-1} & \text{if } a = i+1 \end{cases}$$

where $\tilde{\pi}_j^{(i)}$ is the smallest part in $\tilde{\pi}^{(i)}$ such that $\tilde{\pi}_j^{(i)} > \tilde{\pi}_1^{(i+1)}$. (If we define μ dually, then $\tilde{\mu}^{(i)}$ is the partition obtained from $\tilde{\pi}^{(i)}$ by deleting the part $\tilde{\pi}_j^{(i)}$ and $\tilde{\mu}^{(i+1)}$ is obtained by adding the part $\tilde{\pi}_j^{(i)} - 1$ to $\tilde{\pi}^{(i+1)}$.) In other words, the corresponding module $M(\mu)$ is obtained by deleting the simple top S_i of the indecomposable summand of $M(\pi)$ corresponding to the j th column $\tilde{\pi}_j^{(i)}$ of $\tilde{\pi}^{(i)}$. It is easy to see that μ is separated and $\sigma_i^+(\sigma_i^-\pi) = \pi$.

We now can associate to each $\pi \in \Pi^s$ a word w with $\wp(w) = \pi$ (and so $m = |\pi| := \sum_{i \in I, j \geq 1} \pi_j^{(i)}$). If $\pi = \emptyset$, we set $w = 1$. Otherwise, $\pi \in \Pi_{i_1}^s$ for some i_1 with $\sigma_{i_1}^-(\pi) \in \Pi_{i_2}^s$, etc. So we obtain a sequence of i_1, \dots, i_m such that

$$\sigma_{i_m}^- \cdots \sigma_{i_1}^-(\pi) = \emptyset.$$

Let $w = i_1 \cdots i_m \in \Omega$. Then $\wp(w) = \pi$. Hence, \wp maps onto Π^s . □

Note that, in general, $\sigma_i^-(\sigma_i^+\pi) \neq \pi$. For example, let $n = 2$ and

$$\pi = ((\pi^{(1)} = (2, 1, 1), \pi^{(2)} = (1, 1)).$$

Then $\sigma_1^-(\sigma_1^+\pi) = ((2, 2, 2), \emptyset)$ does not equal to π .

Example 4.2. Let us consider Example 3.6, that is, $n = 3$ and

$$\pi = (\pi^{(1)} = (4, 3, 3, 1, 1), \pi^{(2)} = (3, 2, 1), \pi^{(3)} = (2, 2)).$$

Since $\tilde{\pi}_1^{(1)} = 5, \tilde{\pi}_1^{(2)} = 3, \tilde{\pi}_1^{(3)} = 2$, we have both $\pi \in \Pi_1^s$ and $\pi \in \Pi_2^s$. Then

$$\begin{aligned} \sigma_1^-\pi &= ((3, 2, 2), (4, 3, 2, 1) (2, 2)), \\ \sigma_2^-\pi &= ((4, 3, 3, 1, 1), (2, 1) (3, 3)). \end{aligned}$$

Repeating the above process, we totally get nine words in this way

$$\begin{array}{lll} 12^2 1^2 3^4 2^4 1^5 3^3 2, & 21^3 2^3 4^2 4^1 5^3 3^2, & 21^3 3^2 3^3 2^4 1^5 3^3 2, \\ 21^3 3^2 2^3 2^2 4^1 5^3 3^2, & 21^3 3^3 2^5 3^1 5^3 3^2, & 21^3 3^3 2^5 1^3 1^4 3^3 2, \\ 21^3 3^3 2^5 1^2 3^1 3^3 2, & 21^3 3^3 2^5 1^3 3^1 2^3 3^2, & 21^3 3^3 2^5 1^4 3^4 12. \end{array}$$

However, they are not all the words in the fibre $\wp^{-1}(\pi)$. In fact, $\wp^{-1}(\pi)$ consists of 141 words.

Remark 4.3. In [13, 4.4] Ringel considers a proper subset Ω^c of Ω consisting of condensed words (see the definition in [13, 4.4]) and defines a map $\varepsilon : \Omega^c \rightarrow \Pi^s$. His map ε is in fact the restriction of our map $\wp : \Omega \rightarrow \Pi^s$. This can be seen from [13, 3.4, 5.1, Thm C] and results 6.1 and 6.3 below.

5. DISTINGUISHED WORDS

For each word $w = i_1 i_2 \cdots i_m \in \Omega$ and each representation M in $\mathbb{T}(n, k)$, a composition series

$$M = M_0 \supset M_1 \supset \cdots \supset M_{m-1} \supset M_m = 0$$

of M is said to be of type w if $M_{j-1}/M_j \cong S_{i_j}$ for all $1 \leq j \leq m$. We denote by $\langle w, M \rangle$ the number of composition series of M of type w . Note that if k is an infinite field, $\langle w, M \rangle$ may be infinite.

Lemma 5.1. *Let F be another filed. Then for each $w \in \Omega$ and each $\lambda \in \Pi$, we have*

$$\langle w, M_k(\lambda) \rangle \neq 0 \iff \langle w, M_F(\lambda) \rangle \neq 0.$$

Proof. Let $w = i_1 i_2 \cdots i_m$ and $\langle w, M_k(\lambda) \rangle \neq 0$. We use induction on m to show $\langle w, M_F(\lambda) \rangle \neq 0$.

If $m = 0$ or 1 , this is obvious. Let $m > 1$. Since $\langle w, M_k(\lambda) \rangle \neq 0$, $M_k(\lambda)$ admits a submodule N such that $M_k(\lambda)/N \cong S_{i_1 k}$ and $\langle w', N \rangle \neq 0$, where $w' = i_2 \cdots i_m$. Then there is a unique $\mu \in \Pi$ with $N \cong M_k(\mu)$. Thus, we obtain an exact sequence

$$0 \longrightarrow M_k(\mu) \longrightarrow M_k(\lambda) \longrightarrow S_{i_1 k} \longrightarrow 0.$$

Now Lemma 2.1(2) implies that there exists a similar exact sequence in $\mathbb{T}(n, F)$

$$0 \longrightarrow M_F(\mu) \longrightarrow M_F(\lambda) \longrightarrow S_{i_1 F} \longrightarrow 0.$$

By induction, $\langle w', M_k(\mu) \rangle = \langle w', N \rangle \neq 0$ implies $\langle w', M_F(\mu) \rangle \neq 0$. Consequently, $\langle w, M_F(\lambda) \rangle \neq 0$.

The converse is obtained by reversing the roles of k and F . \square

We recall from [13, 2.3] the definition of a reduced filtration. Let $w = i_1 i_2 \cdots i_m$ be a word in Ω . Then w can be uniquely expressed in the *tight form* $w = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t}$, where $e_r \geq 1 \forall r$, and $j_r \neq j_{r+1}$ for $1 \leq r \leq t-1$. A filtration

$$M = M_0 \supset M_1 \supset \cdots \supset M_{t-1} \supset M_t = 0$$

of a nilpotent representation M is called a *reduced filtration* of type w if $M_{r-1}/M_r \cong e_r S_{j_r}$ for all $1 \leq r \leq t$. Note that any reduced filtration of M of type w can be refined to a composition series of M of type w . Conversely, given a composition series of M of type w , there is a unique reduced filtration of M of type w such that the given composition series is a refinement of this reduced filtration.

Definition 5.2. A word w is called *distinguished* if the module $M_k(\wp(w))$ over an algebraically closed field k has a unique reduced filtration of type w .

Remark 5.3. The condensed words constructed in [13, 4.4] are distinguished words (see [13, 5.8]), but the converse is not true. For example, let $n = 3$ and

$$\pi = (\pi^{(1)} = (3, 2, 1), \pi^{(2)} = (1, 1), \pi^{(3)} = (1)).$$

Then the fibre $\wp^{-1}(\pi)$ contains 7 words:

$$\begin{aligned} &12^21^23^32, \quad 21^33^22^23, \quad 2121^23^32, \quad 21^323^32, \quad 21^33^2232, \\ &21^3323^22, \quad 21^2213^32. \end{aligned}$$

Among them, the first two words are the only condensed ones, but the first five words are distinguished.

In the following we are going to characterize all distinguished words. Recall from §2 the notation $M_i(p) = \bigoplus_{j=1}^m S_i[p_j]$, where p is a partition and $i \in I$. We start with the following lemma.

Lemma 5.4. Let $p = (p_1, p_2, \dots, p_m)$ be a partition with m parts. For a fixed subsequence $1 \leq j_1 < j_2 < \dots < j_t \leq m$, let

$$N = \bigoplus_{r=1}^t S_{i+1}[p_{j_r} - 1] \oplus \bigoplus_{j \neq j_1, \dots, j_t} S_i[p_j].$$

Then $M_i(p)$ has a unique submodule isomorphic to N if and only if $j_r = r$ for all $1 \leq r \leq t$ and $p_t > p_{t+1}$. Moreover, if this is the case and k is algebraically closed, then each submodule N' of $M_i(p)$ with $M_i(p)/N' \cong tS_i$ satisfies $N \leq N'$.

Proof. Suppose that $j_r = r$ for all $1 \leq r \leq t$ and $p_t > p_{t+1}$. It is easy to see that, if U_j denotes the unique maximal submodule of $S_i[p_j]$, the submodule

$$X := \bigoplus_{r=1}^t U_r \oplus \bigoplus_{j=t+1}^m S_i[p_j]$$

is isomorphic to N . Clearly, $M_i(p)/X \cong tS_i$. Let X' be a submodule of $M_i(p)$ isomorphic to N . Then there is a short exact sequence

$$0 \longrightarrow X' \xrightarrow{\iota} M_i(p) \xrightarrow{\phi} tS_i \longrightarrow 0,$$

where ι denotes the inclusion. Choose $v_j \in S_i[p_j] \setminus U_j$. Since $\bigoplus_{r=1}^m U_r = \text{rad}(M_i(p)) \subseteq X'$, Lemma 2.1(3) together with the condition $p_t > p_{t+1}$ implies that $\phi(v_1), \dots, \phi(v_t)$ are linearly independent and that $\phi(v_j) = 0$ for $t+1 \leq j \leq m$. This forces $X' = \text{Ker } \phi = X$.

Conversely, suppose that M has a unique submodule isomorphic to N and suppose that either $j_r > r$ for some $1 \leq r \leq t$ or $j_r = r$ for all $1 \leq r \leq t$, but $p_t = p_{t+1}$. In both cases, there exists an $l \neq j_r$ for $1 \leq r \leq t$ such that $p_l \geq p_{j_t}$. For each $x \in k$, we denote by T_x the submodule of $S_i[p_{j_t}] \oplus S_i[p_l]$ generated by U_{j_t} and $xv_t + v_l$. It is easy to see that T_x are

pairwise distinct and all isomorphic to $S_{i+1}[p_{j_t} - 1] \oplus S_i[p_l]$ with $(S_i[p_{j_t}] \oplus S_i[p_l])/T_x \cong S_i$. For each $x \in k$, we set

$$N_x = \bigoplus_{r=1}^{t-1} U_{j_r} \oplus T_x \oplus \bigoplus_{j \neq l, j_1, \dots, j_t} S_i[p_j]$$

Then $N_x, x \in k$, are pairwise distinct submodules of M which are all isomorphic to N . This is a contradiction. Thus, we have $j_r = r$ for all $1 \leq r \leq t$ and $p_t > p_{t+1}$.

To see the second assertion, let k be an algebraically closed field, and suppose $j_r = r$ for all $1 \leq r \leq t$ and $p_t > p_{t+1}$. Take any submodule N' of $M_i(p)$ such that $M_i(p)/N' \cong tS_i$. By Lemma 2.1(2), there are $1 \leq l_1 < l_2 < \dots < l_t \leq m$ such that

$$N' \cong \bigoplus_{r=1}^t S_{i+1}[p_{l_r} - 1] \oplus \bigoplus_{j \neq l_1, \dots, l_t} S_i[p_j].$$

Let s be the number of l_r 's with $l_r > t$. We use induction on s to prove

$$\bigoplus_{r=1}^t S_{i+1}[p_r - 1] \oplus \bigoplus_{j=t+1}^m S_i[p_j] = N \leq N'.$$

If $s = 0$, then $N' \cong N$. If $s \geq 1$, then $l_t \geq t + 1$ and there exists an $a \leq t$ such that $a \neq l_r$ for all $1 \leq r \leq t$ and $p_a \geq p_t > p_{l_t}$. This inequality $p_a > p_{l_t}$ gives rise to a non-split exact sequence

$$0 \longrightarrow S_{i+1}[p_a - 1] \longrightarrow S_i[p_a] \oplus S_{i+1}[p_{l_t} - 1] \longrightarrow S_i[p_{l_t}] \longrightarrow 0,$$

which implies by Lemma 3.1

$$S_i[p_{l_t}] \oplus S_{i+1}[p_a - 1] < S_i[p_a] \oplus S_{i+1}[p_{l_t} - 1].$$

Thus,

$$N'' := \bigoplus_{r=1}^{t-1} S_{i+1}[p_{l_r} - 1] \oplus S_i[p_{l_t}] \oplus S_{i+1}[p_a - 1] \oplus \bigoplus_{j \neq a, l_1, \dots, l_t} S_i[p_j] < N'.$$

The inductive hypothesis finally shows $N \leq N'' < N'$. This completes the proof. \square

Let $w = i_1^{e_1} i_2^{e_2} \dots i_t^{e_t} \in \Omega$ be in the tight form. For each $0 \leq a \leq t$, we put $w_a = i_{a+1}^{e_{a+1}} \dots i_t^{e_t}$ and $\nu(a) = \wp(w_a)$. In particular, $w_0 = w$ and $w_t = 1$. Further, for $a \geq 1$, we have

$$\nu(a-1) = \wp(w_{a-1}) = \underbrace{\sigma_{i_a}^+ \dots \sigma_{i_a}^+}_{e_a} \nu(a).$$

Theorem 5.5. *Maintain the notation above. The word $w = i_1^{e_1} i_2^{e_2} \dots i_t^{e_t} \in \Omega$ is distinguished if and only if for each $1 \leq a \leq t$,*

$$\widetilde{\nu(a)}_{e_a}^{(i_a+1)} \geq \widetilde{\nu(a)}_1^{(i_a)},$$

that is, the e_a th part of the dual partition of $\nu(a)^{(i_a+1)}$ is greater than or equal to the first part of the dual partition of $\nu(a)^{(i_a)}$.

Proof. We first observe from Lemma 3.7 that for each $1 \leq a \leq t$

$$(5.5.1) \quad M(\nu(a-1)) \cong \underbrace{S_{i_a} * \cdots * S_{i_a}}_{e_a} * M(\nu(a)) \cong (e_a S_{i_a}) * M(\nu(a)).$$

Thus, $M := M(\nu(0))$ admits a filtration

$$(5.5.2) \quad M = M_0 \supset M_1 \supset \cdots \supset M_{t-1} \supset M_t = 0$$

such that $M_a \cong M(\nu(a))$ and $M_{a-1}/M_a \cong e_a S_{i_a} \forall a$. This is a reduced filtration of M of type w .

We claim that a submodule N_a of $M(\nu(a-1))$ with $N_a \cong M(\nu(a))$ is unique if and only if $\widetilde{\nu(a)}_{e_a}^{(i_a+1)} \geq \widetilde{\nu(a)}_1^{(i_a)}$. Indeed, from the definition of $\sigma_{i_a}^+$, we see that $\widetilde{\nu(a-1)}^{(i_a)}$ is obtained by adding parts $\widetilde{\nu(a)}_j^{(i_a+1)} + 1$ ($1 \leq j \leq e_a$) to $\widetilde{\nu(a)}^{(i_a)}$. Thus, by Lemma 5.4, N_a has the described property if and only if

$$\begin{aligned} \widetilde{\nu(a-1)}_j^{(i_a)} &= \widetilde{\nu(a)}_j^{(i_a+1)} + 1, 1 \leq j \leq e_a, \text{ and} \\ \widetilde{\nu(a-1)}_{e_a}^{(i_a)} &> \widetilde{\nu(a-1)}_{e_a+1}^{(i_a)} = \widetilde{\nu(a)}_1^{(i_a)}, \end{aligned}$$

which is equivalent to the condition $\widetilde{\nu(a)}_{e_a}^{(i_a+1)} \geq \widetilde{\nu(a)}_1^{(i_a)}$.

We now prove the theorem by induction on t . If $t = 0$ or 1 , the theorem is obviously true. Let now $t > 1$.

We first assume that w is distinguished. Then the above filtration (5.5.2) is the unique reduced filtration of M of type w . Thus M_1 is the unique submodule of M such that $M_1 \cong M(\nu(1))$ and $M/M_1 \cong e_1 S_{i_1}$. This implies by the claim that $\widetilde{\nu(1)}_{e_1}^{(i_1+1)} \geq \widetilde{\nu(1)}_1^{(i_1)}$. Certainly, the subword w_1 is also distinguished. By induction, we infer that $\widetilde{\nu(a)}_{e_a}^{(i_a+1)} \geq \widetilde{\nu(a)}_1^{(i_a)}$ for $1 \leq a \leq t$.

Conversely, assume that $\widetilde{\nu(a)}_{e_a}^{(i_a+1)} \geq \widetilde{\nu(a)}_1^{(i_a)}$ for all $1 \leq a \leq t$. We also assume that k is an algebraically closed field. Let

$$M = M'_0 \supset M'_1 \supset \cdots \supset M'_{t-1} \supset M'_t = 0$$

be any reduced filtration of M of type w . Then, each M'_i ($0 \leq i \leq t-1$) is an extension of $e_{i+1} S_{i+1}$ by M'_{i+1} . Inductively, we may assume $M'_{i+1} \leq M_{i+1}$. Thus, (5.5.1) and Lemma 3.4 imply $M'_i \leq M_i$. In particular, we have $M'_1 \leq M_1$. On the other hand, since $M/M'_1 \cong e_1 S_{i_1}$, Lemma 5.4 implies $M_1 \leq M'_1$. Hence, $M'_1 \cong M_1 \cong M(\nu(1))$. This implies $M'_1 = M_1$ since by the claim M has a unique submodule isomorphic to $M(\nu(1))$. Now, by induction, w_1 is distinguished, that is, $M_1 \cong M(\nu(1))$ has a unique reduced filtration of type w_1 . Therefore, M has a unique reduced filtration of type w , i.e., w is distinguished. \square

6. A COMPARISON OF ORDER RELATIONS

For $\lambda, \mu \in \Pi$, we write $\mu \leq \lambda$ if $M(\mu) \leq M(\lambda)$. Then \leq defines a partial order on the set Π which is independent of k by Lemma 3.2 (and the remark afterwards). We first prove in this section that this order coincides with the extension order \preceq introduced in §2.

Proposition 6.1. *The order relations \leq and \preceq on Π coincide.*

Proof. For $\lambda, \mu \in \Pi$, it is easy to see from Lemma 3.1 that $\mu \preceq \lambda$ implies $\mu \leq \lambda$. It remains to prove that $\mu \leq \lambda$ implies $\mu \preceq \lambda$. We assume that k is algebraically closed and write $M(\mu) \preceq M(\lambda)$ if $\mu \preceq \lambda$.

Let $\mu \leq \lambda$ in Π and set $N := M(\mu)$, $M := M(\lambda)$. By Lemma 3.2(4), to prove $N \preceq M$, it suffices to show that if there is an exact sequence

$$(6.1.1) \quad 0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0,$$

then $X \oplus Y \preceq M$. We apply induction on $d := \dim_k M + \dim_k Y$. If $d = 0$ or 1 , this is clearly true. Suppose $d > 1$. If Y is decomposable, say $Y = Y' \oplus Y''$ with $Y' \neq 0 \neq Y''$, then we obtain a split exact sequence $0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0$. This together with (6.1.1) gives a commutative diagram as described in Lemma 1.2. In particular, we have exact sequences

$$0 \rightarrow X \rightarrow M' \rightarrow Y' \rightarrow 0 \text{ and } 0 \rightarrow M' \rightarrow M \rightarrow Y'' \rightarrow 0$$

Since both $\dim_k M' + \dim_k Y' < d$ and $\dim_k M + \dim_k Y'' < d$, the inductive hypothesis implies

$$X \oplus Y' \preceq M' \text{ and } M' \oplus Y'' \preceq M.$$

This concludes $X \oplus Y = X \oplus Y' \oplus Y'' \preceq M' \oplus Y'' \preceq M$. Hence, we may suppose that Y is indecomposable. Similarly, X can also be supposed to be indecomposable. In the case where both X and Y are indecomposable, $X \oplus Y \preceq M$ follows from Lemma 2.2. \square

With the coincidence of the two orders, results 5.4 and Theorem C in [13] continue to hold if the order \preceq is replaced by \leq . However, if we use the order \leq at the outset, we may use the nice properties of generic extensions. The following “dominance” property is given in [13, 5.4]. Here we provide a new and short proof.

Lemma 6.2. *Let $w \in \Omega$ and $\lambda \geq \mu$ in Π . Then $\langle w, M(\lambda) \rangle \neq 0$ implies $\langle w, M(\mu) \rangle \neq 0$.*

Proof. In view of Lemma 5.1, we may suppose that k is an algebraically closed field.

Let $w = i_1 i_2 \cdots i_m$ and set $w' = i_2 \cdots i_m$. We apply induction on m . If $m = 1$ then $\lambda \geq \mu$ forces $M(\lambda) = M(\mu)$ and the result is clear. Let now $m > 1$. Since $\langle w, M(\lambda) \rangle \neq 0$, $M(\lambda)$ has a submodule M' such that $\langle w', M' \rangle \neq 0$ and $M(\lambda)/M' \cong S_{i_1}$. Thus, by Prop. 3.4, we obtain $M(\lambda) \leq S_{i_1} * M'$. This together with $M(\mu) \leq M(\lambda)$ yields $M(\mu) \leq S_{i_1} * M'$. Now, applying Prop. 3.4 to this relation, there are modules N', N'' with $N' \leq M'$, $N'' \leq S_{i_1}$, and an exact sequence

$$0 \longrightarrow N' \longrightarrow M(\mu) \longrightarrow N'' \longrightarrow 0.$$

Since S_{i_1} is simple, $N'' \leq S_{i_1}$ implies $N'' \cong S_{i_1}$. On the other hand, the inductive hypothesis implies $\langle w', N' \rangle \neq 0$, that is, N' has a composition series of type w' . Therefore, $M(\mu)$ has a composition series of type w , i.e., $\langle w, M(\mu) \rangle \neq 0$. \square

Theorem C in [13] plays a key role in the proofs of the main results there. It holds only for condensed words and requires a rather long proof. We are now able to generalize this result by removing the restriction on the words.

Theorem 6.3. *For each $w \in \Omega$ and each $\pi \in \Pi$, $\langle w, M(\pi) \rangle \neq 0$ if and only if $\pi \leq \wp(w)$.*

Proof. Again by Lemma 5.1 and Remark 3.8(2), we may assume that k is an algebraically closed field.

Let $w = i_1 i_2 \cdots i_m \in \Omega$, $\pi \in \Pi$, and $\langle w, M(\pi) \rangle \neq 0$. We use induction on m to prove that $\pi \leq \wp(w)$, i.e., $M(\pi) \leq M(\wp(w))$. If $m = 0$, or 1, there is nothing to prove. Let $m > 1$ and set $w' = i_2 \cdots i_m$. Then

$$M(w) = S_{i_1} * (S_{i_2} * \cdots * S_{i_m}) = S_{i_1} * M(w').$$

Since $\langle w, M(\pi) \rangle \neq 0$, $M(\pi)$ has a submodule M' with $M(\pi)/M' \cong S_{i_1}$ and M' has a composition series of type w' . Thus, there is an exact sequence

$$0 \longrightarrow M' \longrightarrow M(\pi) \longrightarrow S_{i_1} \longrightarrow 0.$$

By the inductive hypothesis, we have $M' \leq M(w')$. We infer from Prop. 3.4 that

$$M(\pi) \leq S_{i_1} * M' \leq S_{i_1} * M(w') = M(w) = M(\wp(w)),$$

that is, $\pi \leq \wp(w)$.

Conversely, let $\pi \leq \wp(w)$. By the definition of $\wp(w)$, $\langle w, M(\wp(w)) \rangle \neq 0$. This together with Lemma 6.2 implies $\langle w, M(\pi) \rangle \neq 0$. \square

For $\lambda \in \Pi$, let $\Pi^{\leq \lambda}$ be the poset ideal generated by λ , i.e.,

$$\Pi^{\leq \lambda} = \{\mu \in \Pi \mid \mu \leq \lambda\}.$$

Corollary 6.4. *If $\lambda \in \Pi^s$, then $\mu \in \Pi^{\leq \lambda}$ if and only if there exists a $w \in \wp^{-1}(\lambda)$ such that $M(\mu)$ has a composition series of type w .*

It would be interesting to describe the ideal $\Pi^{\leq \lambda}$ for an arbitrary $\lambda \in \Pi$.

7. HALL ALGEBRAS AND ITS COMPOSITION SUBALGEBRA

Given three modules L, M, N in \mathbb{T} , let F_{MN}^L be the number of submodules V of L such that $V \cong N$ and $L/V \cong M$. More generally, given modules M, N_1, \dots, N_m in \mathbb{T} , we let $F_{N_1 \dots N_m}^M$ be the number of the filtrations

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{m-1} \supseteq M_m = 0,$$

such that $M_{t-1}/M_t \cong N_t$ for all $1 \leq t \leq m$. In particular, we have, for $w = i_1 \cdots i_m \in \Omega$, $F_{S_{i_1} \dots S_{i_m}}^M = \langle w, M \rangle$. Note that

$$(7.0.1) \quad F_{N_1 N_2 N_3}^M = \sum_{[N]} F_{N_1 N}^M F_{N_2 N_3}^N.$$

By [13] and [5], there exist Hall polynomials in \mathbb{T} . More precisely, for π, μ_1, \dots, μ_m in Π , there is a polynomial $\varphi_{\mu_1 \dots \mu_m}^\pi(q) \in \mathcal{A} := \mathbb{Z}[q]$ such that for any finite k of q_k elements

$$\varphi_{\mu_1 \dots \mu_m}^\pi(q_k) = F_{M_k(\mu_1) \dots M_k(\mu_m)}^{M_k(\pi)}.$$

In particular, when $M(\mu_j) = S_{i_j}$ for all j , we denote the polynomial $\varphi_{\mu_1 \dots \mu_m}^\pi$ by $\langle w | \pi \rangle$ as in [13, 8.1].

Lemma 7.1. *Let $w \in \Omega$ be a distinguished word with the tight form $i_1^{e_1} i_2^{e_2} \cdots i_t^{e_t}$. Then*

$$\langle w | \wp(w) \rangle = \prod_{a=1}^t \llbracket e_a \rrbracket^!,$$

where $\llbracket e_a \rrbracket^! = \llbracket 1 \rrbracket \cdots \llbracket e_a \rrbracket$ with $\llbracket m \rrbracket = \frac{1-q^m}{1-q} \in \mathcal{A}$.

Proof. We first observe that, since every composition series of type w is a refinement of the associated reduced filtration of type w ,

$$\langle w, M(w) \rangle = F_{e_1 S_{i_1} \cdots e_t S_{i_t}}^{M(w)} \prod_{a=1}^t \langle i_a^{e_a}, e_a S_{i_a} \rangle.$$

If w is distinguished, it follows that $M(w)$ has a unique reduced filtration for an infinite number of finite fields. Thus $F_{e_1 S_{i_1} \cdots e_t S_{i_t}}^{M(w)} \equiv 1$ (cf. Lemma 5.1). Now the result follows from the fact that the number of composition series of $e_a S_{i_a}$ is $\llbracket e_a \rrbracket^!$ (cf. [13, 8.2]). \square

The generic (untwisted) *Ringel-Hall algebra* $\mathcal{H}_q(\Delta)$ of Δ is by definition the free \mathcal{A} -module with basis $\{u_\pi | \pi \in \Pi\}$, and the multiplication is given by

$$u_\mu u_\nu = \sum_{\pi \in \Pi} \varphi_{\mu\nu}^\pi(q) u_\pi.$$

In practice, we sometimes write $u_\pi = u_{[M(\pi)]}$ in order to make certain calculations in term of modules. For $i \in I$, we set $u_i = u_{[S_i]}$ and denote by $\mathcal{C}_q(\Delta)$ the subalgebra of $\mathcal{H}_q(\Delta)$ generated by u_i , $i \in I$. This is called the (generic) *composition algebra* of Δ . It is easy to see that $\mathcal{C}_q(\Delta)$ is a proper subalgebra of $\mathcal{H}_q(\Delta)$. Moreover, both $\mathcal{H}_q(\Delta)$ and $\mathcal{C}_q(\Delta)$ admit a natural \mathbb{N}^n -grading by dimension vectors.

Let $\mathcal{A}\Omega$ be the semigroup algebra of Ω , which is indeed the free \mathcal{A} -algebra generated by the set I . Thus, we have an algebra homomorphism $f : \mathcal{A}\Omega \rightarrow \mathcal{C}_q(\Delta)$ with $f(i) = u_i$ for all $i \in I$.

Following [13, 1.2], there is an \mathcal{A} -bilinear form

$$\langle - | - \rangle : \mathcal{A}\Omega \times \mathcal{H}_q(\Delta) \rightarrow \mathcal{A}, \quad (w, u_\pi) \mapsto \langle w | \pi \rangle(q).$$

We denote by \mathfrak{R} the set of $x \in \mathcal{A}\Omega$ such that $\langle x | - \rangle = 0$. We call \mathfrak{R} the *left radical* of the form.

For each word $w = i_1 i_2 \cdots i_m \in \Omega$, we define

$$u_w = u_{i_1} u_{i_2} \cdots u_{i_m} \in \mathcal{C}_q(\Delta).$$

Lemma 7.2. *We have $\mathfrak{R} = \ker(f)$. Thus f induces an isomorphism $\mathcal{C}_q(\Delta) \cong \mathcal{A}\Omega/\mathfrak{R}$. Moreover, $\mathcal{C}_q(\Delta) \otimes_{\mathbb{Z}[q]} \mathbb{Q}[q]$ is free.*

Proof. Define the \mathcal{A} -bilinear form

$$(-, -) : \mathcal{H}_q(\Delta) \times \mathcal{H}_q(\Delta) \rightarrow \mathcal{A}$$

by $(u_\lambda, u_\mu) = \delta_{\lambda\mu}$. Then $\langle w | \pi \rangle = (u_w, u_\pi)$. Thus, for any $x = \sum_w x_w w \in \mathcal{A}\Omega$, we have

$$\langle x | \pi \rangle = \sum_w x_w (u_w, u_\pi) = (f(x), u_\pi).$$

Therefore, the equality follows. \square

For any commutative ring \mathcal{A}' which is an \mathcal{A} -algebra and any \mathcal{A} -module M , let $M_{\mathcal{A}'} = M \otimes_{\mathcal{A}} \mathcal{A}'$ denote the \mathcal{A}' -module obtained from M by base change to \mathcal{A}' . Clearly, base change induces an \mathcal{A}' -bilinear form

$$\langle - | - \rangle_{\mathcal{A}'} : \mathcal{A}'\Omega \times \mathcal{H}_q(\Delta)_{\mathcal{A}'} \rightarrow \mathcal{A}'.$$

It is easy to see that, if \mathcal{A}' is an integral domain containing \mathcal{A} , then $\mathfrak{R}_{\mathcal{A}'}$ is the same as the left radical of $\langle - | - \rangle_{\mathcal{A}'}$.

Ringel discovered that the ideal \mathfrak{R} contains the ideal \mathfrak{R}' generated by those near quantum Serre relations (see [13, 8.6]) and that these two ideals are equal after base change to the localization $\mathcal{A}_{(q-1)} = \mathbb{Z}[q]_{(q-1)} = \mathbb{Q}[q]_{(q-1)}$ of \mathcal{A} at the maximal ideal generated by $q - 1$. Thus, we have the following theorem (see [13, 8.7]).

Theorem 7.3. *Let $\mathcal{A}' = \mathcal{A}_{(q-1)}$.*

(1) *If $n \geq 3$, then $\mathcal{C}_q(\Delta_n)_{\mathcal{A}'}$ is generated by u_i , $i \in I$, with relations:*

$$\begin{aligned} u_i u_j &= u_j u_i \quad \text{if } j \not\equiv i \pm 1 \pmod{n}, \quad i, j \in I, \\ u_i^2 u_{i+1} - (q+1) u_i u_{i+1} u_i + q u_{i+1} u_i^2 &= 0, \\ u_i u_{i+1}^2 - (q+1) u_{i+1} u_i u_{i+1} + q u_{i+1}^2 u_i &= 0, \quad i \in I. \end{aligned}$$

(2) *The algebra $\mathcal{C}_q(\Delta_2)_{\mathcal{A}'}$ is generated by u_1 and u_2 with relations:*

$$\begin{aligned} q u_1^3 u_2 - (q^2 + q + 1) u_1^2 u_2 u_1 + (q^2 + q + 1) u_1 u_2 u_1^2 - q u_2 u_1^3 &= 0, \\ q u_2^3 u_1 - (q^2 + q + 1) u_2^2 u_1 u_2 + (q^2 + q + 1) u_2 u_1 u_2^2 - q u_1 u_2^3 &= 0. \end{aligned}$$

The following results generalize [13, 8.5, 8.8].

Theorem 7.4. *For every $\pi \in \Pi^s$, choose an arbitrary word $w_\pi \in \wp^{-1}(\pi)$.*

(1) *The free \mathcal{A} -submodule \mathfrak{C} of $\mathcal{A}\Omega$ spanned by $\{w_\pi | \pi \in \Pi^s\}$ intersects \mathfrak{R} trivially.*

(2) *We have $\mathbb{Q}(q)\Omega = \mathfrak{C}_{\mathbb{Q}(q)} \oplus \mathfrak{R}_{\mathbb{Q}(q)}$.*

(3) *Let $\mathcal{A}' = \mathcal{A}_{(q-1)}$ and assume that all w_π are distinguished. Then $\mathcal{A}'\Omega = \mathfrak{C}_{\mathcal{A}'} \oplus \mathfrak{R}_{\mathcal{A}'}$.*

Proof. If $x = \sum_{i=1}^m a_i w_i \in \mathfrak{C} \cap \mathfrak{R}$, then $\wp(w_i) \neq \wp(w_j)$ for $i \neq j$. It follows from Theorem 6.3 that the matrix $(\langle w_i | \wp(w_j) \rangle)$ is upper triangular under an appropriate ordering and the diagonal entries are non-zero. This implies $x = 0$, proving (1).

To prove (2) resp. (3), it remains to prove that $\mathbb{Q}(q)\Omega = \mathfrak{C}_{\mathbb{Q}(q)} + \mathfrak{R}_{\mathbb{Q}(q)}$ resp. $\mathcal{A}'\Omega = \mathfrak{C}_{\mathcal{A}'} + \mathfrak{R}_{\mathcal{A}'}$. This can be done in a way similar to the proof of [13, 8.8], using Theorems 6.3 and 7.3. Note that, if w_π is distinguished, then, by Lemma 7.1, $\langle w | \wp(w) \rangle$ is invertible in \mathcal{A}' . This is just the case considered by Ringel. \square

An immediate consequence is the following monomial basis theorem for the composition algebra $\mathcal{C}_q(\Delta)_{\mathcal{A}'}$.

Corollary 7.5. *For every $\pi \in \Pi^s$, choose an arbitrary word $w_\pi \in \wp^{-1}(\pi)$. The set $\{u_{w_\pi} | \pi \in \Pi^s\}$ is a $\mathbb{Q}(q)$ -basis of $\mathcal{C}_q(\Delta)_{\mathbb{Q}(q)}$. Moreover, if all w_π are chosen to be distinguished, then this set is an \mathcal{A}' -basis of $\mathcal{C}_q(\Delta)_{\mathcal{A}'}$ where $\mathcal{A}' = \mathcal{A}_{(q-1)}$.*

8. PROOF OF THEOREM 1.1

We now in this section transfer the monomial bases for the composition algebra of a cyclic quiver to a quantum affine \mathfrak{sl}_n . We need to consider the twisted version of Hall and composition algebras (see [14]).

First, we recall that the Euler form associated with the cyclic quiver Δ is the bilinear form $\varepsilon(-, -) : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}$ defined by

$$\varepsilon(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^n a_i b_i - \sum_{i=1}^n a_i b_{i+1},$$

where $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$, and $b_{n+1} = b_1$. It is well-known that for two representations $M, N \in \mathbb{T}$, there holds

$$\varepsilon(\dim M, \dim N) = \dim_k \operatorname{Hom}(M, N) - \dim_k \operatorname{Ext}^1(M, N).$$

Let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$, where v is an indeterminate with $v^2 = q$. The *twisted Ringel-Hall algebra* $\mathcal{H}_v^*(\Delta)$ of Δ is by definition the free \mathcal{Z} -module with basis $\{u_\pi = u_{[M(\pi)]} | \pi \in \Pi\}$, and the multiplication is defined by

$$u_\mu \star u_\nu = v^{\varepsilon(\mu, \nu)} \sum_{\pi \in \Pi} \varphi_{\mu\nu}^\pi(v^2) u_\pi,$$

where $\varepsilon(\mu, \nu) = \varepsilon(\dim M(\mu), \dim M(\nu))$. The \mathcal{Z} -subalgebra $\mathcal{C}_v^*(\Delta)$ of $\mathcal{H}_v^*(\Delta)$ generated by u_i , $i \in I$, is called the *twisted composition algebra*.

Let U^+ be the \mathcal{Z} -subalgebra of $\mathbf{U} = \mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ generated by $E_i, i \in I$, and let $\mathcal{Z}' = \mathcal{Z}_{(v-1)}$ be the localization of $\mathbb{Z}[v, v^{-1}]$ at the ideal generated by $v - 1$. Since the relations in Theorem 7.3 become the quantum Serre relations under the twisted multiplication, we obtain by modifying Ringel's proof of [13, 8.7] an isomorphism

$$\Phi : \mathcal{C}_v^*(\Delta)_{\mathcal{Z}'} \cong U_{\mathcal{Z}'}^+, \quad u_i \mapsto E_i, \quad i \in I.$$

For each $w = i_1 i_2 \cdots i_m \in \Omega$, if we put

$$E_w = E_{i_1} E_{i_2} \cdots E_{i_m} \in U^+$$

as in the introduction, we have the following (cf. Theorem 7.5).

Theorem 8.1. *For every $\pi \in \Pi^s$, choose an arbitrary word $w_\pi \in \wp^{-1}(\pi)$. The set $\{E_{w_\pi} | \pi \in \Pi^s\}$ is a $\mathbb{Q}(v)$ -basis of \mathbf{U}^+ . Moreover, if all w_π are chosen to be distinguished, then this set is a \mathcal{Z}' -basis of $U_{\mathcal{Z}'}^+$.*

This theorem together with the triangular decomposition of \mathbf{U} gives Theorem 1.1.

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the affine Lie algebra $\widehat{\mathfrak{sl}}_n$ over \mathbb{Q} of type \tilde{A}_{n-1} with generators e_i, f_i, h_j . Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra of \mathfrak{g} . We also define monomials e_w similarly for $w \in \Omega$ in $\mathfrak{U}(\mathfrak{n}_+)$. Then, we have the following.

Corollary 8.2. *For every $\pi \in \Pi^s$, choose an arbitrary distinguished word $w_\pi \in \wp^{-1}(\pi)$. The set $\{e_{w_\pi} | \pi \in \Pi^s\}$ is a \mathbb{Q} -basis of $\mathfrak{U}(\mathfrak{n}_+)$.*

Proof. By Theorem 7.3, we have $\mathcal{C}_q(\Delta)_{\mathcal{A}'} / (q-1)\mathcal{C}_q(\Delta)_{\mathcal{A}'} \cong \mathfrak{U}(\mathfrak{n}_+)$. The result follows from Theorem 7.5. \square

9. A COMPARISON OF BASES

With the isomorphism Φ above, we now identify \mathbf{U}^+ with $\mathcal{C}_v^*(\Delta)_{\mathbb{Q}(v)}$. Thus, $E_i = u_i$ and, for each $w = i_1 i_2 \cdots i_m \in \Omega$, we have

$$E_w = u_w^* := u_{i_1} \star u_{i_2} \star \cdots \star u_{i_m} = v^{\varepsilon(w)} u_w,$$

where $\varepsilon(w) = \sum_{r \leq t} \varepsilon(\dim S_{i_r}, \dim S_{i_t})$. Further, we have the following relation between monomial bases $\{E_{w_\pi}\}_{\pi \in \Pi^s}$ and the defining basis $\{u_\lambda\}_{\lambda \in \Pi}$ of $\mathcal{H}_v^*(\Delta)$.

Proposition 9.1. *For each $w \in \Omega$, we have*

$$(9.1.1) \quad E_w = \sum_{\lambda \leq \varphi(w)} v^{\varepsilon(w)} \langle w | \lambda \rangle u_\lambda.$$

Moreover, the coefficients appearing in the sum are all non-zero and all in $\mathbb{N}[v, v^{-1}]$.

Proof. By Theorem 6.3, it remains to show that each polynomial $\langle w | \lambda \rangle \in \mathbb{Z}[q]$ has non-negative coefficients.

Let $w = i_1 i_2 \cdots i_m$. If $m = 1$ or 2 , $\langle w | \lambda \rangle \in \mathbb{Z}[q]$ has obviously the required property. Assume $m \geq 2$ and set $w' = i_2 \cdots i_m$. Then we have by (7.0.1)

$$\langle w | \lambda \rangle = \sum_{\mu} \varphi_{\nu\mu}^\lambda(q) \langle w' | \mu \rangle,$$

where $\nu \in \Pi$ satisfies $M(\nu) = S_{i_1}$. By induction, it suffices to prove that each $\varphi_{\nu\mu}^\lambda(q)$ lies in $\mathbb{N}[q]$. Let k be a finite field of q_k elements. We now calculate the number $f := F_{M_k(\nu)M_k(\mu)}^{M_k(\lambda)}$. Since $M_k(\nu) = S_{i_1}$, we have clearly $f = F_{S_{i_1}N}^M$ with

$$M = M_{i_1}(\tilde{\lambda}^{(i_1)}) = \oplus_j S_{i_1}[\tilde{\lambda}_j^{(i_1)}] \text{ and } N = M_{i_1}(\tilde{\mu}^{(i_1)}).$$

For simplicity, we write $i = i_1$, $p := \tilde{\lambda}^{(i_1)} = (p_1, \dots, p_l)$. By Lemma 2.1(2), $F_{S_{i_1}N}^M \neq 0$ implies $N \cong N_t$ for some $1 \leq t \leq l$, where $N_t = S_{i+1}[p_t - 1] \oplus \bigoplus_{j:j \neq t} S_i[p_j]$. Let a (resp. b , c) be the number of p_j 's such that $p_j > p_t$ (resp. $p_j = p_t$, $p_t > p_j$) so that $l = a + b + c$. We claim that

$$f = F_{S_{i_1}N_t}^M = q_k^a (1 + q_k + \cdots + q_k^{b-1}) = q_k^a \llbracket b \rrbracket.$$

Indeed, choose $v_j \in S_i[p_j] \setminus \text{rad}(S_i[p_j])$ for each $1 \leq j \leq l$. Then (the images of) v_1, \dots, v_l form a basis for $M/\text{rad}(M)$. Let V denote the subspace of M spanned by v_{a+1}, \dots, v_{a+b} and let Ξ be the set of subspaces of V of codimension 1. Then $|\Xi| = \llbracket b \rrbracket$. For each $W \in \Xi$ and $(x_1, \dots, x_a) \in k^a$, we fix an element $v_W \in V \setminus W$ and define $N(W, x_1, \dots, x_a)$ to be the submodule of M generated by $v_1 + x_1 v_W, \dots, v_a + x_a v_W, W, v_{a+b+1}, \dots, v_l$ and $\text{rad}(M)$. Clearly, the modules $N(W, x_1, \dots, x_a)$ are all distinct submodules isomorphic to N_t and each submodule N of M with $N \cong N_t$ is of this form. Therefore, $f = |\Xi \times k^a| = q_k^a \llbracket b \rrbracket$, as required. Consequently, each $\varphi_{\nu\mu}^\lambda(q)$ lies in $\mathbb{N}[q]$. \square

We observe that the relation (9.1.1) shares similar properties possessed by the canonical basis $\{b_\pi\}_{\pi \in \Pi}$ introduced in [8]. It is known (cf. [6, (8)]) that, when writing b_π as a linear

combination of the defining basis $\{u_\lambda\}_{\lambda \in \Pi}$, both order and positivity properties are satisfied. More precisely, we have

$$(9.1.2) \quad b_\pi = \sum_{\lambda \leq \pi} p_{\lambda, \pi} u_\lambda, \quad p_{\lambda, \pi} \in \mathbb{N}[v, v^{-1}].$$

However, there is no elementary proof for the positivity property in this case. Note that the subset $\{b_\pi\}_{\pi \in \Pi^s}$ forms a basis for \mathbf{U}^+ . It is also observed (see [6, p.40]) that non-separated π may occur in the right hand side of (9.1.2) (and of (9.1.1) as well).

Motivated by [13, Thm 4], we now construct another basis for \mathbf{U}^+ from the defining basis $\{u_\lambda\}_{\lambda \in \Pi}$. Recall from §7 the $\mathbb{Q}(v)$ -bilinear form $\langle - | - \rangle : \mathbb{Q}(v)\Omega \times \mathcal{H}_q(\Delta)_{\mathbb{Q}(v)} \rightarrow \mathbb{Q}(v)$. Let \mathfrak{S} denote the right radical of this form, that is, the space of all $y \in \mathcal{H}_q(\Delta)_{\mathbb{Q}(v)}$ with $\langle - | y \rangle = 0$. Obviously, $\mathfrak{S} = \oplus_{\mathbf{d}} \mathfrak{S}_{\mathbf{d}}$ admits the grading by dimension vectors. By [13, Thm 4], we see that \mathfrak{S} is a direct complement of the subspace of $\mathcal{H}_q(\Delta)_{\mathbb{Q}(v)}$ spanned by all $u_\pi, \pi \in \Pi^s$. However, in contrast to the left radical \mathfrak{R} , \mathfrak{S} is in general not an ideal of $\mathcal{H}_q(\Delta)_{\mathbb{Q}(v)}$. For example, let $n = 2$. Then $\mathfrak{S}_{(1,1)}$ is one-dimensional over $\mathbb{Q}(v)$ with a basis element $x = -u_\lambda + u_\mu + u_\nu$, where $\lambda = ((1), (1))$, $\mu = ((1, 1), \emptyset)$, and $\nu = (\emptyset, (1, 1))$ are pairs of partitions in Π . But an easy calculation shows that neither $u_1 x$ nor $x u_1$ lies in $\mathfrak{S}_{(2,1)}$. Thus \mathfrak{S} is not an ideal of $\mathcal{H}_q(\Delta_2)_{\mathbb{Q}(v)}$.

In order to obtain an ideal, we use Green's form [4] which is a modified version of the bilinear form $\langle - | - \rangle$. We consider the twisted version $\mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)}$ and $\mathcal{C}_v^*(\Delta)_{\mathbb{Q}(v)} = \mathbf{U}^+$.

For each $\pi \in \Pi$, it is well-known that there is a monic polynomial $a_\pi \in \mathbb{Z}[q]$ such that $a_\pi(q_k) = |\text{Aut}(M_k(\pi))|$ for any finite field k of q_k elements. Define a $\mathbb{Q}(v)$ -bilinear form

$$\langle - | - \rangle' : \mathbb{Q}(v)\Omega \times \mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)} \longrightarrow \mathbb{Q}(v)$$

by setting $\langle w | u_\pi \rangle' = v^{\varepsilon(w)} a_\pi^{-1} \langle w | u_\pi \rangle$, and denote by \mathfrak{S}' the right radical subspace of this form. Let $\mathbb{Q}(v)\Pi^s$ be the subspace of $\mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)}$ spanned by all $u_\pi, \pi \in \Pi^s$.

Lemma 9.2. *The subspace \mathfrak{S}' is an ideal of $\mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)}$ and moreover*

$$\mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)} = \mathbb{Q}(v)\Pi^s \oplus \mathfrak{S}'.$$

Proof. Following [4], there is a symmetric and non-degenerate bilinear form

$$(-, -)' : \mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)} \times \mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)} \longrightarrow \mathbb{Q}(v)$$

defined by $(u_\lambda, u_\mu)' = a_\lambda^{-1} \delta_{\lambda\mu}$. Then $\langle w | u_\pi \rangle' = (u_w^*, u_\pi)'$. This implies

$$\mathfrak{S}'_{\mathbf{d}} = \{y \in (\mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)})_{\mathbf{d}} \mid ((\mathcal{C}_v^*(\Delta)_{\mathbb{Q}(v)})_{\mathbf{d}}, y)' = 0\}$$

for each $\mathbf{d} \in \mathbb{N}^n$. By [4, Thm 1], we see easily that \mathfrak{S}' is an ideal of $\mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)}$.

Now, by an argument similar to the proof of [13, Thm 4], we see that $\mathbb{Q}(v)\Pi^s \cap \mathfrak{S}' = 0$. On the other hand, since $(-, -)'$ is non-degenerate and its restriction to $\mathcal{C}_v^*(\Delta)_{\mathbb{Q}(v)}$ is again non-degenerate (see [4] and [9, 1.2, 33.1]), we obtain $(\mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)})_{\mathbf{d}} = (\mathcal{C}_v^*(\Delta)_{\mathbb{Q}(v)})_{\mathbf{d}} \oplus \mathfrak{S}'_{\mathbf{d}}$. Thus,

$$\dim \mathfrak{S}'_{\mathbf{d}} = \dim (\mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)})_{\mathbf{d}} - \dim (\mathcal{C}_v^*(\Delta)_{\mathbb{Q}(v)})_{\mathbf{d}} = |\Pi_{\mathbf{d}}| - |\Pi_{\mathbf{d}}^s|,$$

where $\Pi_{\mathbf{d}} = \{\pi \in \Pi \mid \dim M(\pi) = \mathbf{d}\}$ and $\Pi_{\mathbf{d}}^s = \Pi^s \cap \Pi_{\mathbf{d}}$. Hence,

$$\mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)} = \mathbb{Q}(v)\Pi^s \oplus \mathfrak{S}'.$$

□

Theorem 9.3. *We have $\mathbf{U}^+ \cong \mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)}/\mathfrak{S}'$. Moreover, the algebra \mathbf{U}^+ has a basis of the form $\mathcal{P} := \{\bar{u}_\pi \mid \pi \in \Pi^s\}$ where \bar{u}_π is the homomorphic image of u_π .*

Proof. By the decomposition $\mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)} = \mathcal{C}_v^*(\Delta)_{\mathbb{Q}(v)} \oplus \mathfrak{S}'$, the required isomorphism is given by the composition of the canonical inclusion $\mathcal{C}_v^*(\Delta)_{\mathbb{Q}(v)} \hookrightarrow \mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)}$ and the projection $\mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)} \twoheadrightarrow \mathcal{H}_v^*(\Delta)_{\mathbb{Q}(v)}/\mathfrak{S}'$. The rest of the proof follows from the previous lemma. \square

Remark 9.4. The basis \mathcal{P} given in Theorem 9.3 may be viewed as a PBW type basis. We remark that the transition matrix from any monomial basis (or the canonical basis) to the basis \mathcal{P} is no longer in triangular form. That is, if we rewrite the right hand side of (9.1.1) or (9.1.2) by replacing those u_π with $\pi \notin \Pi^s$ by linear combinations of elements of \mathcal{P} , then the order relation is no longer sustained.

Example 9.5. Consider the case $n = 2$. Then $\Omega_{(2,1)}$ consists of three words $w_1 = 1^2 2$, $w_2 = 21^2$, $w_3 = 121$, and $\Pi_{(2,1)}$ contains four elements

$$\begin{aligned}\alpha &= ((2), (1)), \beta = ((2, 1), \emptyset), \\ \gamma &= ((1), (1, 1)), \delta = ((1, 1, 1), \emptyset),\end{aligned}$$

which are ordered by $\alpha < \beta, \alpha < \gamma, \beta < \delta$ and $\gamma < \delta$. Clearly, $\wp(w_1) = \beta, \wp(w_2) = \gamma$ and $\wp(w_3) = \delta$ are separated. Further, $\mathfrak{S}'_{(2,1)}$ is one-dimensional with a basis element $-(v^4 - 1)u_\alpha + u_\beta + u_\gamma + u_\delta$, that is, in $\mathbf{U}_{(2,1)}^+$,

$$\bar{u}_\alpha = \frac{1}{v^4 - 1}(\bar{u}_\beta + \bar{u}_\gamma + \bar{u}_\delta),$$

and $\mathbf{U}_{(2,1)}^+$ has bases $\{E_{w_1}, E_{w_2}, E_{w_3}\}$ and $\{\bar{u}_\beta, \bar{u}_\gamma, \bar{u}_\delta\}$ satisfying

$$\begin{aligned}E_{w_1} &= \frac{1}{v(v^2-1)}(v^4\bar{u}_\beta + \bar{u}_\gamma + \bar{u}_\delta), \\ E_{w_2} &= \frac{1}{v(v^2-1)}(\bar{u}_\beta + v^4\bar{u}_\gamma + \bar{u}_\delta), \\ E_{w_3} &= \frac{v}{v^2-1}(\bar{u}_\beta + \bar{u}_\gamma + \bar{u}_\delta).\end{aligned}$$

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